Equivalence

DEFINITION

'Equivalent' is the technical mathematical term that corresponds to the everyday language, 'is the same as'. In learning mathematics, children often encounter situations in which they have to recognize that all the things in a given set are in some sense 'the same'. In doing this they have therefore identified an equivalence. This process of recognizing samenesses in sets of mathematical entities is an important way of thinking in mathematics.

EXPLANATION AND DISCUSSION

Recognizing similarities and differences is a fundamental cognitive process by means of which we organize and make sense of our experiences. This process is significant in most fields of human activity, from social interaction to science. It has a particular application in learning mathematics, in terms of equivalence (what is the same?) and transformation (what is different?). Some examples will demonstrate how widespread is the concept of equivalence in the experience of children learning mathematics, both in terms of conceptual learning and as a tool for manipulating mathematical ideas.

In the early stages of number, children learn to recognize that there is something the same about, say, a set of three chocolates and a set of three teddy bears. These two collections are different from each other (teddy bears are different from chocolates), but there is something very significantly the same about them. They are both described by the adjective 'three', which describes a property they share. The fact that they share this property of 'threeness' can be shown by one-to-one matching, a dynamic experience for the young child that makes what is the same about three chocolates and three teddy bears explicit. The property that makes these two sets equivalent is then abstracted from many collections of three things to form the concept of 'three' as a cardinal number, existing in its own right, independent of any specific context.

This example illustrates how many abstract mathematical concepts are formed by identifying equivalences, recognizing things that are the same or properties that are shared. Geometry provides many such examples. In learning the concept of 'square', for example, children may engage in sorting a set of two-dimensional shapes that includes a number of squares of different sizes. When they put together all the squares into a subset, because they are 'all the same shape', they are using reasoning on the basis of equivalence. The shapes are not all the same, of course, because they may differ in size or colour, for example. But they are all the same in some sense; they share a property; there is an equivalence.

This is one reason why this kind of thinking is so powerful and fundamental to mathematics. It enables learners to hold in their mind one conceptual idea (such as 'three' or 'square') which is an abstraction of their experiences of many specific examples of the concept, all of which are in some sense the same, all of which are equivalent in this respect. In doing this, the learner combines a number of individual experiences of specific exemplars, which have been recognized as being the same in some sense, into one abstraction.

An equivalence relation when applied to a set of mathematical entities partitions that set into a number of subsets, within which the items are 'equivalent', that is, they share a mathematical property. For example, if children investigate the remainders that arise when the integers from 1 to 20 are divided by 3, they find there are three subsets, with numbers that give remainders 0, 1 and 2:

1. {3, 6, 9, 12, 15, 18};
2. {1, 4, 7, 10, 13, 16, 19};
3. {2, 5, 8, 11, 14, 17, 30}.

Within each of these three subsets the numbers have a shared property.

1. They are all multiples of 3.
2. They are all 1 more than a multiple of 3.
3. They are all 2 more than a multiple of 3.

Technically, the three subsets are called 'equivalence classes' (Williams and Shuard, 1994: 495–7). In learning mathematics, children often sort mathematical entities into equivalence classes. For example, sorting a set of two-dimensional shapes into those with the same numbers of sides generates these equivalence classes: triangles, quadrilaterals, pentagons, hexagons, and so on.

The concept of equivalence is also powerful as a mathematical tool. If two mathematical objects are regarded in some sense as the same, then, in appropriate circumstances, one member of an equivalence class can be replaced by any other. For example, fractions can be sorted into sets of equivalent fractions, such as {$^1/_4$, $^2/_8$, $^3/_{12}$, $^4/_{16}$, … } The fraction $^2/_8$ is not in every sense the same as the fraction $^1/_4$: one piece of a cake cut into four equal parts is not identical in every respect to two pieces of the cake cut into 8 equal parts. However, it is the same amount of cake. There is an equivalence here, which pupils have to learn to recognize – and then to use. For example, when adding $^3/_8$ to $^1/_4$, it is appropriate and helpful to be able to replace the $^1/_4$ by the equivalent fraction $^2/_8$.

Primary school teachers should therefore be aware of the importance of promoting this kind of thinking in pupils, getting them to identify and to use equivalences. Every opportunity should be taken to ask the basic question, 'What is the same?' Pupils should be asked to look at pairs or larger collections of numbers, shapes and other mathematical entities and identify ways in which they are the same, or to sort them into subsets of items that share some property. Alongside this, pupils can be asked to say in what ways two numbers, shapes or objects are different: this task engages the pupil in the corresponding process of transformation.

PRACTICAL EXAMPLES

Four examples below illustrate the prevalence of the concept of equivalence in primary mathematics.

One–one matching

Young children are given one-to-one matching experience by setting a table for six people, with each place-setting having one knife, one fork, one plate, one cup, one saucer, one spoon. The teacher and children together count how many of each item there are on the table, and then arrange them in different ways to make the matching explicit. The teacher repeatedly uses phrases such as 'one of these for one of those' and 'the number of these is the same as the number of those'.

Shapes: What's my rule?

The teacher and a group of children look together at a set of three-dimensional geometric shapes. The teacher tells the children that there is a rule for sorting these shapes. If a shape satisfies the rule it goes into the 'yes' group, otherwise into the 'no' group. After a few examples of shapes being placed into one or other of the two groups, the children are asked to predict whether the remaining shapes are yes or no. When the sorting is completed, the children are asked to say what is the same about all the shapes; what is the rule the teacher has used? This is repeated with different rules. Then the children may have the opportunity to use their own rules to sort the shapes. Each time this is done the 'yes' group demonstrates an equivalence. For example, if the rule used is 'all the angles are right angles', then the set of equivalent shapes selected is the set of cuboids.

Numbers: What's my rule?

This is a similar activity with integers (whole numbers). The teacher asks the children to suggest numbers and then writes these on the board in either the 'yes' set or the 'no' set. The children have to guess the rule and say what is the same about all the numbers in the 'yes' set. Examples of rules might be 'between 9 and 100' (the set of 2-digit numbers) or 'the final digit is 6 or 1' (all these numbers are 1 more than a multiple of 5).

What is the same?

Pupils are given two shapes (such as those in Figure 12) or two numbers (such as 16 and 36) and invited to find as many ways as they can in which the two numbers or shapes are the same. They formulate sentences beginning with the words, 'they both … ' or 'they are both … '. For example, for the shapes, 'they both have four sides'. Or, for the numbers, 'they are both square numbers'.

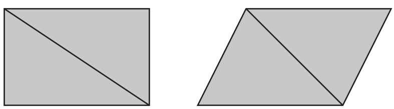

Figure 12 *What is the same about these two shapes?*

FURTHER READING

Equivalence is one of the central themes of Haylock and Cockburn (2003), featuring particularly in chapter 7 in their analysis of children's learning about shape and space.